NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2887

ON THE STABILITY OF THE LAMINAR MIXING REGION BETWEEN TWO PARALLEL STREAMS IN A GAS

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Washington
January 1953

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SUMMARY

The stability of the mixing of two streams was studied both for the interest in the problem and for clarifying certain points in the basic stability theory. It is shown that, when the relative speed of the two parallel streams exceeds the sum of their velocities of sound, subsonic oscillations cannot occur and the mixing region may be expected to be stable with respect to small disturbances. It is further shown that, when viscosity and heat conductivity are neglected, if the flow can execute a small neutral subsonic oscillation of finite wave length, it can also execute self-excited oscillations of longer wave lengths and damped oscillations of shorter wave lengths.

Rigorous developments of the mathematical theory of asymptotic solutions confirm previous methods of solution of the stability equations in a compressible fluid. This theory also shows that, at high Reynolds numbers, the damped oscillations in a strictly parallel main flow have a structure similar to that of the vorticity field in fully developed turbulent flow.

Sample calculations are also included exhibiting various quantitative properties of these small oscillations.

INTRODUCTION

The mixing of two parallel streams of gas occurs in a number of cases. An interesting example is furnished by the slip stream in a three-shock configuration. It has long been suggested that such laminar mixing zones could, at sufficiently high speeds, be stable with respect to small disturbances although they are known to be very unstable at low speeds. The purpose of the present investigation is to find out some of these stability characteristics. Apart from the development of the general theory, there are included the calculations of the neutral and unstable oscillations, the extent of the amplification, and other related properties.

The basic equations for the study of small disturbances in the laminar boundary layer of gases have been given in reference 1. However, the development of the theory there has specific reference to the case of a layer near a solid boundary. As pointed out in references 2 and 3, the stability theory for a mixing zone in an incompressible fluid differs from that for a layer near a solid boundary in that solutions of the exponential type are unimportant. This leads to the conclusion that the effects of viscosity and heat conduction are negligible, except at very low Reynolds numbers, in determining the characteristics of the oscillations. To confirm this point, a rigorous mathematical theory of asymptotic solutions was developed for the compressible case similar to that indicated in reference 4 for the incompressible case. In view of the mathematical interest involved, it was decided that this basic part of the present investigation would be published separately in mathematical journals (see references 5 and 6), and only the main results and their physical significance will be presented here.

As in the case of an incompressible fluid, the "inviscid" case is expected to be characteristic of the behavior of the disturbances at moderately large Reynolds numbers. Most of the studies are, therefore, made in the inviscid case. However, the interpretation of the inviscid case must be subjected to the same care as in the incompressible case; that is, in the case of damped disturbances, the differential equation of the inviscid flow may not be regarded as valid throughout the real axis. There is a finite viscous region even in the limit of vanishing viscosity. The complex conjugate of the amplified disturbance is certainly a solution of the inviscid equation, but it is not a limiting solution of the complete viscous equation. This behavior of the inviscid solution reminds one of the vorticity structure of fully developed turbulent flow as found by Batchelor and Townsend (reference 7).

The nonexistence of subsonic disturbances is usually associated with the stability of the parallel flow. There seems to be some basis for doing this, although the role of supersonic disturbances has never been fully clarified. It is easy to see that, for certain combinations of the properties of the two streams, it is impossible to have a subsonic disturbance relative to both. Under such conditions, one may expect stability. These conditions for stability are developed herein and are expected to hold, irrespective of the viscous effects.

Applying the theory of stability in the inviscid case, one can further narrow down the possible range of instability. This will depend upon the velocity and temperature distributions in the shear zone. In the present work, calculations are made for gases with Prandtl number equal to unity. Although the condition of equal total enthalpy in the two streams is also used, it is shown that this restriction can be immediately removed by considering a moving observer. It is found that the condition of stability thus found does not differ very much

from that found above from general considerations. Thus, it may be surmised that the exact distributions of temperature and velocity have only a secondary influence on the stability characteristics in the mixing zone. Thus, the approximations used in the present calculations of the basic velocity and temperature distributions cannot influence the final results to any appreciable extent.

The neutral disturbances are of two kinds: (1) A steady deviation and (2) an oscillation of finite wave length. Thus, there are two branches of the neutral curve at infinite Reynolds numbers. They may be expected to join together at low Reynolds numbers enclosing a region of instability. Calculations of neutral and amplified oscillations are carried out in a number of cases with one stream at rest.

This investigation, carried out at the Massachusetts Institute of Technology, was sponsored by and conducted with the financial assistance of the National Advisory Committee for Aeronautics. The author is indebted to Mr. D. W. Dunn for his valuable suggestions and help in the preparation of the final version of the report and to Miss Diana Mason and Mr. W. V. Caldwell for their help in making the numerical calculations.

STEADY FLOW IN THE LAMINAR LAYER BETWEEN TWO PARALLEL STREAMS

The basic steady flow under discussion is a boundary-layer flow with no body forces and no pressure gradient. The basic equations are (see list of symbols in the appendix):

$$\rho^* u^* \frac{\partial u^*}{\partial x^*} + \rho^* v^* \frac{\partial u^*}{\partial y^*} = \frac{\partial}{\partial y^*}\left(\mu^* \frac{\partial u^*}{\partial y^*}\right) \tag{1}$$

$$\frac{\partial}{\partial x^*}(\rho^* u^*) + \frac{\partial}{\partial y^*}(\rho^* v^*) = 0 \tag{2}$$

$$\rho^* u^* \frac{\partial}{\partial x^*}\left(c_p T^*\right) + \rho^* v^* \frac{\partial}{\partial y^*}\left(c_p T^*\right) = \frac{\partial}{\partial y^*}\left(k^* \frac{\partial T^*}{\partial y^*}\right) + \mu^*\left(\frac{\partial u^*}{\partial y^*}\right)^2 \tag{3}$$

the pressure being a constant throughout the field. In the case of a homogeneous incompressible fluid, with the streams at the same temperature, the temperature may be taken as constant throughout the field, and the first two equations can be solved for the velocity distributions $u^*(x^*,y^*)$ and $v^*(x^*,y^*)$. In the case of a compressible gas, the integration has to be carried out for individual cases. However, if the Prandtl number $c_p\mu^*/k^*$ is equal to unity, it is known that there is a quadratic relation of the type

$$c_pT^* + \frac{1}{2}u^{*2} = A + Bu^* \tag{4}$$

between the temperature and the velocity, and one is again essentially dealing with two distributions $u^*(x^*,y^*)$ and $v^*(x^*,y^*)$. Indeed, the constants A and B are given in terms of the conditions in the parallel streams as follows:

$$\left.\begin{aligned} B &= \frac{1}{2}\left(U_2 + U_1\right) + c_p\left(T_2 - T_1\right)\Big/\left(U_2 - U_1\right) \\ A &= -\frac{1}{2}U_1U_2 + c_p\left(T_1U_2 - T_2U_1\right)\Big/\left(U_2 - U_1\right) \end{aligned}\right\} \tag{5}$$

If the total enthalpy in the two streams 1 and 2 is the same, that is, if

$$c_pT_1 + \frac{1}{2}U_1^{\,2} = c_pT_2 + \frac{1}{2}U_2^{\,2} \tag{6}$$

then $B = 0$, and the total enthalpy is constant throughout the whole field:

$$c_pT^* + \frac{1}{2}u^{*2} = C \tag{7}$$

This is a particularly simple case, to which, however, all other cases can be easily reduced. This is done by rewriting equation (4) into the form

$$c_pT^* + \frac{1}{2}(u^* - B)^2 = A + B^2/2 \tag{8}$$

One need only consider an observer moving with the speed B and consider the relative velocity $u^* - B$. Thus, the solution for general specified values of U_1,T_1, and U_2,T_2 can be derived from the isoenergetic solution with boundary conditions $U_1 - B,T_1$ and $U_2 - B,T_2$ by simply adding the constant B to the u^*-component of the velocity.[1]

In the following discussions, isoenergetic basic solutions will be referred to often; however, it should be kept in mind that by the consideration of a moving observer the general case may be obtained. This transformation is not restricted to the steady flow but applies to the consideration of the disturbances as well. Thus, if all the cases of constant total enthalpy are calculated, all the other cases are also known.

So far, the viscosity coefficient may depend on the absolute temperature in any manner. If there is direct proportionality of these quantities, the solution in the compressible case can be expressed in terms of that in the incompressible case. These relations are well-known and, in the following discussion, only the results relevant to this case will be given.

Incompressible Case

For the incompressible case:

$$\left.\begin{aligned} u^* &= U_1f'(\eta) \\ v^* &= U_1\frac{1}{2}\sqrt{\nu_1/U_1x^*}(\eta f' - f) \end{aligned}\right\} \tag{9}$$

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[1]It is to be noticed that in certain cases with $T_2/T_1 < 1$ the boundary value $U_2 - B$ for the corresponding isoenergetic problem may be negative while $U_1 - B$ is still positive. Thus the isoenergetic problem may not be physically significant. However, for the purposes of theoretical analysis, this point is not important.

where

$$\eta = y^* \Big/ \sqrt{\nu_1 x^* / U_1} \tag{10}$$

and $f(\eta)$ satisfies the differential equation

$$ff'' + 2f''' = 0 \tag{11}$$

with the conditions

$$\left.\begin{aligned} f'(\eta) &\longrightarrow 1 \quad \text{as} \quad \eta \longrightarrow \infty \\ f'(\eta) &\longrightarrow \frac{U_2}{U_1} \quad \text{as} \quad \eta \longrightarrow -\infty \end{aligned}\right\} \tag{12}$$

A third condition is arbitrary up to a translation along the η-axis.

For a typical scale, the momentum-boundary-layer thickness θ^* may be introduced, which is defined by

$$\rho_1 U_1{}^2 \theta^* = \int_{-\infty}^{\infty} \rho^* \left(U_1 - u^*\right)\left(u^* - U_2\right) dy^* \tag{13}$$

Then

$$\theta = \frac{\theta^*}{\sqrt{\nu_1 x^* / U_1}} = \int_{-\infty}^{\infty} \left[1 - f'(\eta)\right] \left[f'(\eta) - \frac{U_2}{U_1}\right] d\eta \tag{14}$$

for an incompressible fluid. It will be seen later that the same formulas apply for a compressible fluid in the isoenergetic case.

Calculations made by Görtler (reference 8) for the turbulent mixing region can be easily adopted for the purpose at hand. There is only a

slight difference in the method of representation. To convert his function $F'(\xi)$ to the notation of this report, the following relations should be used

$$\left.\begin{aligned} f'(\eta) &= \frac{Pr}{1 + \lambda} F'(\xi) \\ \eta &= 2\sqrt{1 + \lambda}\,\xi \end{aligned}\right\} \tag{15a}$$

where

$$\lambda = \frac{U_1 - U_2}{U_1 + U_2} \tag{15b}$$

The converted results are given in table I and figure 1.

Compressible Case

For the compressible case with constant total enthalpy, $\mu^* \propto T^*$, where

$$u^* = U_1 f'(\zeta) \tag{16}$$

and

$$T^* = T_1\left(1 + \frac{\gamma - 1}{2} M_1{}^2\right)\left[1 - f'^2(\zeta)\right] \tag{17}$$

with

$$\eta = \int_0^\zeta \left\{1 - \frac{\gamma - 1}{2} M_1{}^2\left[1 - f'^2(\zeta)\right]\right\} d\zeta \tag{18}$$

where

$$\eta = y^*\Big/\sqrt{\nu_1 x^*/U_1}$$

The momentum thickness θ^* can be defined by the same equation (equation (13)) and can be shown to have the same value (equation (14)) in dimensionless form. Velocity and temperature distributions in the case $U_2 = 0$ have been calculated for several Mach numbers; these are tabulated in table II and plotted in figure 2.

A GENERAL RESTRICTION FOR EXISTENCE OF NEUTRAL SUBSONIC DISTURBANCES

Before going into the general theory of stability, a preliminary discussion of the stability of the mixing zone will be given. It will be found below that the results thus obtained give quite an adequate description of the general dependence of the stability of the mixing zone on the Mach numbers of the streams. It will be shown that, if the average Mach number $\overline{M}$ of the relative motion (defined by equations (25) and (26)) exceeds the value 2, subsonic disturbances in the sense of reference 1 cannot exist, and the mixing zone may be expected to be stable.

Consider two parallel streams at speeds U_1 and U_2, temperatures T_1 and T_2, and Mach numbers M_1 and M_2. For definiteness, take $U_1 - U_2 > 0$. Consider an observer moving with the speed U_2. Then the streams appear to have speeds $U_1 - U_2$ and 0, while their temperatures are obviously not changed. If

$$c' = c - U_2 \tag{19}$$

denotes the speed (relative to the moving observer) of a wavy motion propagating in the direction of the stream (c' positive or negative), then the conditions for subsonic disturbances are

$$\left|\left(U_1 - U_2\right) - c'\right| < a_1 \tag{20}$$

$$\left|c'\right| < a_2 \tag{21}$$

where a_1 and a_2 are the velocities of sound in the streams. It is clear that subsonic disturbances can always exist if $U_1 - U_2$ is less

than a_1. For $U_1 - U_2 > a_1$, condition (20) can be satisfied only with $c' > 0$. Thus

$$0 < c' < a_2, \text{ that is, } 0 < c - U_2 < a_2 \tag{22}$$

Now, note that condition (20) is actually

$$\left|U_1 - c\right| < a_1$$

and hence

$$-a_1 < U_1 - c < a_1 \tag{23}$$

Adding relations (22) and (23), it is found that

$$U_1 - U_2 < a_1 + a_2 \tag{24}$$

is a necessary condition for the existence of subsonic disturbances. By introducing the average velocity of sound

$$\overline{a} = \frac{1}{2}\left(a_1 + a_2\right) \tag{25}$$

and the average Mach number of relative motion

$$\overline{M} = \left(U_1 - U_2\right)\Big/\overline{a} \tag{26}$$

defined with respect to this average sound velocity, it may be concluded that subsonic disturbances cannot exist (and the mixing zone may be expected to be stable) if the average Mach number $\overline{M}$ of the relative motion exceeds the value 2, as stated at the beginning of this section.

In terms of the Mach numbers of the two streams, condition (24) states that if

$$M_1 > 1 + \sqrt{T_2/T_1}\left(1 + M_2\right) \tag{27}$$

Subsonic disturbances cannot exist, and the motion is stable. (See fig. 3.)

In the case of constant total enthalpy, the temperature ratio T_2/T_1 can be expressed as

$$\frac{T_2}{T_1} = \frac{1 + (\gamma - 1)M_1^{\,2}/2}{1 + (\gamma - 1)M_2^{\,2}/2} \tag{28}$$

Then condition (27) becomes

$$M_1 > 1 + \left[\frac{1 + (\gamma - 1)M_1^{\,2}/2}{1 + (\gamma - 1)M_2^{\,2}/2}\right]^{1/2}\left(1 + M_2\right) \tag{29}$$

The critical condition is

$$M_1 = 1 + \left[\frac{1 + (\gamma - 1)M_1^{\,2}/2}{1 + (\gamma - 1)M_2^{\,2}/2}\right]^{1/2}\left(1 + M_2\right) \tag{30}$$

Removing the square root,

$$\frac{1 + (\gamma - 1)M_1^{\,2}/2}{1 + (\gamma - 1)M_2^{\,2}/2} = \frac{\left(M_1 - 1\right)^2}{\left(M_2 + 1\right)^2}$$

This form suggests an obvious solution, $M_1 = -M_2$. Another solution can then be easily obtained as

$$M_1 = \frac{(3 - \gamma)M_2 + 4}{(3 - \gamma) - 2(\gamma - 1)M_2} \tag{31}$$

By substituting in equation (30), $M_1 = -M_2$ is found to be an extraneous solution. The only solution of equation (30) is then given by equation (31).

Thus the flow is stable if

$$M_1 > \frac{(3 - \gamma)M_2 + 4}{(3 - \gamma) - 2(\gamma - 1)M_2} \tag{32}$$

The curve for equation (31) is shown in figure 4, where the region to the left of curve A is the region of possible instability. Note the symmetry with respect to the line $M_1 = -M_2$, corresponding to a change of the positive direction of the x-axis in the physical problem. The line $M_1 = M_2$ is drawn to take care of the condition $U_1 - U_2 > 0$, since $U_1 = U_2$ when $M_1 = M_2$. The asymptotes of the curve for equation (31) are

$$M_2 = (3 - \gamma)/2(\gamma - 1) \tag{33}$$

and

$$M_1 = -(3 - \gamma)/2(\gamma - 1) \tag{34}$$

With $\gamma = 1.4$, these values are ± 2. The intercepts are

$$M_1 = 4/(3 - \gamma) \tag{35}$$

and

$$M_2 = -4/(3 - \gamma) \tag{36}$$

With $\gamma = 1.4$, these values are ± 2.5. (Cf. reference 9.)

GENERAL STUDY OF SMALL DISTURBANCES IN A NEARLY PARALLEL FLOW FIELD IN A COMPRESSIBLE FLUID

The general theory of a small disturbance in a field of nearly parallel flow of a gas has been developed in reference 1. The rigorous mathematical proof and improvement of the theory are given in detail in references 5 and 6. In this report, merely the main conclusions and their physical interpretations are outlined without going into the details. Applications of the theory to the specific case at hand will be discussed in some detail.

Consider a nearly parallel stream with dimensionless velocity and temperature distributions $w(y)$ and $T(y)$. The neglect of the dependence of these quantities on x and the omission of the y-component of the basic flow can be justified by detailed investigations. The linearized differential equations for small disturbances then possess solutions of the type

$$\left.\begin{aligned} u' &= \mathrm{Re}\left[f(y)e^{i\alpha(x-ct)}\right] \\ v' &= \mathrm{Re}\left[\alpha\phi(y)e^{i\alpha(x-ct)}\right] \\ \rho' &= \mathrm{Re}\left[r(y)e^{i\alpha(x-ct)}\right] \\ p' &= \mathrm{Re}\left[\pi(y)e^{i\alpha(x-ct)}\right] \\ T' &= \mathrm{Re}\left[\theta(y)e^{i\alpha(x-ct)}\right] \end{aligned}\right\} \tag{37}$$

where u', v', ρ', p', and T' are the perturbations of the two components of velocity, the density, the pressure, and the temperature, all in a suitably defined dimensionless form. The constants α and c are, respectively, the real wave number and the complex wave speed.

The differential equations for the amplitude functions $f(y)$, $\phi(y)$, $r(y)$, $\pi(y)$, and $\theta(y)$ are rather complicated. However, if the effects of viscosity and heat conduction are neglected, they become a relatively simple system, which can be reduced to the following single differential equation for $\phi(y)$:

$$\frac{d}{dy}\left[\frac{(w - c)\phi' - w'\phi}{T - M_1^2(w - c)^2}\right] - \frac{\alpha^2(w - c)\phi}{T} = 0 \tag{38}$$

All the other variables can be expressed in terms of ϕ and ϕ' as follows:

$$\left.\begin{aligned}
f &= -i\left[M^2(w - c)w'\phi - T\phi'\right]\Big/\left[T - M^2(w - c)^2\right] \\
r &= i\left[(\phi' + if) + \rho'\phi\right]\Big/(w - c) \\
\pi &= i\gamma M^2\rho\left[i(w - c)f + w'\phi\right] \\
\theta &= T(\pi/p - r/\rho)
\end{aligned}\right\} \tag{39}$$

The boundary conditions are that the disturbance should be bounded as $y \longrightarrow \pm\infty$.

The inviscid system would have given a well-defined characteristic-value problem if it were not for the fact that differential equation (38) has a singularity at the point $y = y_c$ where $w(y) = c$. This singularity disappears only when $d(\rho w')/dy = 0$ at the same point. Otherwise, a solution of equation (38) has a logarithmic singularity at $y = y_c$ and the characteristic-value problem associated with this equation becomes indeterminate until the proper branch of the solution is determined.

The determination of the proper branch of the solution and its associated physical interpretation is one of the most delicate points in the theory of hydrodynamic stability. The mathematical analysis of the solutions of the complete viscous equations and their limiting solutions will be made first before discussing their physical interpretation.

The complete system of viscous equations can be shown to be equivalent to a system of six linear equations of the first order in six unknowns. Thus, there are six independent solutions. These solutions have been formally obtained as asymptotic series in reference 1, and their rigorous mathematical investigation has been carried out in

references 5 and 6. It is found that two of the six solutions can be expressed in asymptotic series of the form

$$\left.\begin{aligned} f &= f^{(0)} + \frac{1}{\lambda^2} f^{(1)} + \ldots \\ \phi &= \phi^{(0)} + \frac{1}{\lambda^2} \phi^{(1)} + \ldots \end{aligned}\right\} \tag{40}$$

where $f^{(0)}$, $\phi^{(0)}$, . . . are the inviscid solutions satisfying equations (38) and (39) and $\lambda^2 = \alpha R$, where R is the Reynolds number based on the thickness of the mixing region. Thus, the formal limit of equations (40) does approach the inviscid solution, but a complete study of these equations also carries the knowledge of the proper branch to be used.

Four other solutions of the complete system of viscous equations are of the form

$$\left.\begin{aligned} f &= F \exp\left(\lambda Q_i\right) \\ \phi &= \Phi \exp\left(\lambda Q_i\right) \end{aligned} \qquad i = 1, 2, 3, 4 \right\} \tag{41}$$

where

$$\left.\begin{aligned} Q_1 = -Q_2 &= \int_{y_c}^{y} \sqrt{\frac{i}{\nu} (w - c)}\, dy \\ Q_3 = -Q_4 &= \int_{y_c}^{y} \sqrt{\frac{iPr_o}{\nu} (w - c)}\, dy \end{aligned}\right\} \tag{42}$$

and F and Φ can be expressed as power series of λ, involving only a finite number of positive powers.

In the establishment of these asymptotic solutions, it is shown that the lines $\mathrm{Re}\left(Q_1\right) = 0$, $\mathrm{Re}\left(Q_3\right) = 0$, and $\mathrm{Re}\left(Q_1 - Q_3\right) = 0$ are of interest. The geometry of these lines relative to the point y_c (where $w = c$) and the real axis of the y-plane are shown in figure 5. There are asymptotic solutions expressed in equations (40) and (41) which maintain the same analytical expression on the two sides of the dotted lines. However, in crossing the solid lines, they generally change their behavior. Thus, the following conclusion may be drawn: The proper branch of the multiple-valued asymptotic solutions is obtained by taking a path in the complex y-plane below the point $y = y_c$ (in this case $w(y)$ is monotonically increasing along the real axis).

This is the branch taken in reference 1. By examining the behavior of solutions of the type given in equations (41), it can be easily shown that they diverge for either positive or negative large values of y. These solutions should, therefore, be rejected in the present problem. The effect of viscosity is then to be obtained through solutions of the form of equations (40).

GENERAL STABILITY CHARACTERISTICS IN THE INVISCID CASE

Much of the discussion of the inviscid case in reference 1 applies to the present case. However, as noted above, in the present problem, the possibility can be more readily realized that subsonic disturbances may not exist at all, and that the motion may then be expected to be completely stable with respect to small disturbances. Another main difference lies in the "steady" deviation, that is, solutions of equation (38) with $\alpha = 0$. In the present discussion, the general line of discussion in reference 1 will be followed. However, the difference in the boundary conditions often causes a difference in the method of analysis. The arguments are, therefore, presented in some detail. Also, it will be shown that disturbances having wave-lengths slightly longer than that of the neutral subsonic disturbances of finite wave length are unstable while those with slightly shorter wave lengths are the stable ones. This conclusion also applies to the case of the boundary layer, but it was not obtained in reference 1. The analysis also leads to an approximate estimation of the dependence of amplification on wave length. This will be used for the calculation of the amplification of the disturbances in the section "Self-Excited Oscillations."

Some general analytical properties of the solutions of equation (38) will be first summarized, particularly for the case of subsonic disturbances. In this case,

$$T - M_1^2(w - c)^2 > 0 \tag{43}$$

for both free streams. For large positive values of y, equation (38) may be approximated by

$$\phi'' - \beta_1^2\phi = 0 \quad \text{with} \quad \beta_1^2 = \alpha^2\left[1 - M_1^2(1 - c)^2\right] \tag{44}$$

For large negative values of y, it may be approximated by

$$\phi'' - \beta_2^2\phi = 0 \quad \text{with} \quad \beta_2^2 = \alpha^2\left[1 - M_1^2\left(\frac{U_2}{U_1} - c\right)^2 \Big/ T_2\right] \tag{45}$$

Since both β_1^2 and β_2^2 are positive, the solution $\phi(y)$ is exponential in nature for large values of y in the case of subsonic disturbances, with the exception of the case $\alpha = 0$. In that case, two independent solutions of equation (38) are

$$\Phi_1 = w - c \tag{46}$$

and

$$\Phi_2 = (w - c)\int\left[\frac{T}{(w - c)^2} - M_1^2\right] dy \tag{47}$$

The first solution is bounded while the second varies linearly with y for large values of y.

For any value of c, the bounded solution (equation (46)) corresponds to no disturbance at infinity. In fact, for $\alpha = 0$, the disturbance v' is identically zero, by equation (37). The other components of the disturbance are given by equations (39), and it can be easily verified that they all vanish at large distances.

It can be shown from the general nature of the temperature and velocity distributions that condition (43) holds throughout the mixing zone if it holds in both free streams. It is then obvious that

equation (38) has a singularity only at $w - c = 0$. Actually, even if $T - M_1^2(w - c)^2 = 0$, it only gives rise to an apparent singularity.

Analysis of the solution in the neighborhood of the point $y = y_c$ (where $w = c$) gives the following two solutions:

$$\left.\begin{aligned} \phi_1 &= (y - y_c)g_1(y - y_c) \\ \phi_2 &= g_2(y - y_c) + K\phi_1 \log_e (y - y_c) \end{aligned}\right\} \tag{48}$$

where

$$K = \frac{T_c^{\,2}}{(w_c')^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c \tag{49}$$

and g_1 and g_2 are power series in $(y - y_c)$ with $g_1(0) = w_c' \neq 0$ and $g_2(0) = T_c/w_c' \neq 0$. The proper branch of the logarithmic function is to be taken in accordance with the method discussed in the preceding section.

For real values of c, it can be shown that the Reynolds shear stress

$$\tau = -\rho\overline{u'v'} \tag{50}$$

is a constant except for a possible jump at $y = y_c$. In fact, this jump is

$$[\tau] = \frac{\alpha}{2}\,\pi K(w_c')^2\left|\phi_c\right|^2/T_c^{\,2} \tag{51}$$

when one passes from $y_c - 0$ to $y_c + 0$. Thus, for a neutral disturbance with $\alpha \neq 0$, the condition

$$K = 0 \tag{52}$$

must be satisfied, since $\tau = 0$ for $y \longrightarrow \pm\infty$ and, therefore, does not have any jump.

Thus, for the existence of a subsonic disturbance, the quantity,

$$\frac{d}{dy}\left(\frac{1}{T}\frac{dw}{dy}\right) = \frac{d}{dy}\left(\rho\frac{dw}{dy}\right) \tag{53}$$

must vanish at some point in the field; furthermore, the corresponding value w_s of w must be subsonic relative to both streams; that is,

$$1 - \frac{1}{M_1} < w_s < \frac{U_2}{U_1}\left(1 + \frac{1}{M_2}\right) \tag{54}$$

The above reasoning can be applied to the case of the boundary layer; a somewhat different argument was used in reference 1. A little calculation will show that, in general, condition (53) will be satisfied, although condition (54) may not. If the latter is also satisfied, then there actually exists a subsonic disturbance with $c = w_s$. To prove this, equation (38) is rewritten in the form

$$\frac{d}{dy}\left(h\frac{d\phi}{dy}\right) - \left(q + \frac{\alpha^2}{T}\right)\phi = 0 \tag{55}$$

where

$$\left.\begin{aligned} h(y) &= \left[T - M_1^2(w - c)^2\right]^{-1} > 0 \\ q(y) &= \frac{1}{w - c}\frac{d}{dy}\left(h\frac{dw}{dy}\right) \end{aligned}\right\} \tag{56}$$

With $c = w_s$, $q(y)$ is regular all along the real axis. This is a characteristic-value problem for the parameter $k = -\alpha^2$ and is associated with the variational principle

$$\delta\int_{-\infty}^{\infty}\left[h\left(\frac{d\phi}{dy}\right)^2 + q\phi^2\right] dy = 0 \tag{57}$$

with

$$\int_{-\infty}^{\infty} \frac{\phi^2}{T}\, dy = \text{Constant} \tag{58}$$

The least value for $-\alpha^2$ is then given by the minimum value of the ratio

$$k_1 = \int_{-\infty}^{\infty} \left[h\left(\frac{df}{dy}\right)^2 + qf^2\right] dy \Bigg/ \int_{-\infty}^{\infty} \frac{f^2}{T}\, dy \tag{59}$$

for all functions $f(y)$ such that the integrals in this ratio are convergent. Now, it is necessary only to show that for certain functions the ratio of the integrals is negative.

In the choice of such a function, it is necessary for only the numerator in equation (59) to be convergent. The convergence of the integral in the denominator is immaterial, for one can always modify the function at sufficiently large values of $|y|$ so that the denominator becomes convergent without altering the sign of the numerator. To choose a function so that the integral

$$I = \int_{-\infty}^{\infty} (hf'^2 + qf^2)\, dy \tag{60}$$

is negative, it is first noted that

$$I = 0 \quad \text{for} \quad f = w - c \tag{61}$$

This follows from the fact that $\phi = w - c$ is a solution of equation (55) when $\alpha = 0$. It can also be directly verified. Obviously, the value of I is not changed if f is now changed to $|w - c|$. A further modification of the function would yield the desired result. In the neighborhood of y_c, $|w - c|$ is small, but w' is finite. If in this neighborhood the function $f(y) = |w - c|$ is replaced by a horizontal straight line, the integrand in equation (60) is certainly decreased (since $\xi > 0$ and a change in f' causes a larger change in the integral than that in f), and the resultant integral is negative. This completes the proof desired.

Thus, the sufficiency of conditions (53) and (54) for the existence of subsonic disturbances is also established. Let the corresponding value of α be denoted by α_s.

Next, it will be shown that for α slightly less than α_s the disturbances are unstable. To prove this the quantity $(dc/dk)_s$ will be calculated, and its imaginary part will be shown to be positive.

Consider a characteristic function $\phi(y;k,c)$ of equation (55) corresponding to a given value of k. As c changes, k also changes. If equation (55) is differentiated with respect to k, the following equation is obtained

$$\frac{d}{dy}\left(h\frac{d\dot{\phi}}{dy}\right) - q\dot{\phi} + \frac{k}{T}\dot{\phi} + \left[\frac{d}{dy}\left(\frac{\partial h}{\partial c}\frac{d\phi}{dy}\right) - \frac{\partial q}{\partial c}\phi\right]\frac{dc}{dk} + \frac{\phi}{T} = 0 \tag{62}$$

where

$$\dot{\phi} = \frac{\partial\phi}{\partial k} + \frac{\partial\phi}{\partial c}\frac{dc}{dk} \tag{63}$$

Now, multiply equation (62) by ϕ, equation (55) by $\dot{\phi}$, subtract the results, and then integrate with respect to y along a path in the complex plane which leads from $y = -\infty$ on the real axis to $y = \infty$ on the real axis but passes below the point y_c. The following equation is then obtained for dc/dk:

$$\frac{dc}{dk}\int_{-\infty}^{\infty}\left[\frac{\partial h}{\partial c}\left(\frac{d\phi}{dy}\right)^2 + \frac{\partial q}{\partial c}\phi^2\right]dy = \int_{-\infty}^{\infty}\frac{\phi^2}{T}\,dy \tag{64}$$

So far equation (64) is general. Now, specialize to the case of the neutral disturbance in question. Most of the integration can then be carried out along the real axis, with real resultant values. However, the integral

$$J = \int_{-\infty}^{\infty}\frac{\partial q}{\partial c}\phi^2\,dy \tag{65}$$

must be evaluated along an indented path with a small detour below y_c, since $\partial q/\partial c$ has a pole at that point. Calculations show that the imaginary part of J is

$$\operatorname{Im}(J) = \pi\left(\phi_c/w_c{}'\right)^2\left[\frac{d^2}{dy^2}\left(\frac{1}{T}\frac{dw}{dy}\right)\right]_c \tag{66}$$

if this quantity does not vanish. Indeed, this quantity can be seen to be negative. Thus, equation (64) yields a relation of the form

$$\frac{dc}{dk}(P + iQ) = R > 0$$

where $Q = \operatorname{Im}(J)$. Thus,

$$\frac{dc}{dk} = \frac{R}{P^2 + Q^2}(P - iQ) \tag{67}$$

has a positive imaginary part, as required.

In the incompressible case, equation (64) reduces to

$$\frac{dc}{dk}\int_{-\infty}^{\infty}\frac{w''}{(w - c)^2}\phi^2\,dy = \int_{-\infty}^{\infty}\phi^2\,dy \tag{68}$$

and the imaginary part of dk/dc is

$$\operatorname{Im}(dk/dc) = \pi\left(\phi_c\big/w_c{}'\right)^2 w_c{}''' \bigg/ \int_{-\infty}^{\infty}\phi^2\,dy \tag{69}$$

NEUTRAL OSCILLATIONS

From the discussions of the last section, another limitation is imposed on the occurrence of neutral subsonic disturbances. Since the wave speed of the neutral disturbance must be equal to the flow speed w_s at the point where

$$\frac{d}{dy}\left(\rho \frac{dw}{dy}\right) = 0 \tag{70}$$

subsonic disturbances cannot occur if the speed w_s is supersonic relative to either stream. Calculations of w_s are made for the case of uniform total enthalpy. Instead of using equation (70) directly, it is found convenient to transform it into the form

$$\frac{\partial^2}{\partial\psi^2}\left[1 + \frac{2}{(\gamma - 1)M_1{}^2} - w^2\right]^{-1} = 0 \quad \text{with} \quad \psi = f(\zeta) \quad \text{and} \quad w = f'(\zeta) \tag{71}$$

where ψ is the stream function, since the relation $w(\psi)$ is identical with that in the incompressible case. The results of these calculations are shown in table III. In comparing these values against condition (54), it is found that the condition

$$1 - \frac{1}{M_1} < w_s$$

is never violated, but the values of w_s below the solid horizontal lines in the table are too high to satisfy the condition

$$w_s < \frac{U_2}{U_1}\left(1 + \frac{1}{M_2}\right)$$

Thus, for a given Mach number in the stream M_2, the Mach number M_1 in the other stream can become so high that the speed of flow corresponding to condition (70) becomes supersonic relative to the slower free stream. Subsonic disturbances then cannot exist.

The critical case of sonic disturbance is reached when

$$w_s = \frac{U_2}{U_1}\left(1 + \frac{1}{M_2}\right)$$

Calculations for this case are tabulated in table IV. The corresponding values of M_1 and M_2 are plotted as curve B in figure 4 to mark the limit of stability. Only the region to the left of this curve can have subsonic disturbances. It is seen that the condition is more restrictive than that obtained from general considerations alone. In particular, in the case where one stream is at rest, the curve B shows that the flow becomes completely stable at a free-stream Mach number of 1.7, in contrast to the value 2.5 given by curve A. Since they are, however, not very much different, it may be surmised that the exact distributions of temperature and velocity in the mixing region may not be too important in determining the over-all stability characteristics.

It may be recalled that the restriction to the case of constant total enthalpy can be immediately removed by considering a moving observer. Thus, for all cases with Prandtl number equal to unity, it is necessary only to convert the values of the speeds U_1, U_2, and w_s involved. Condition (70) is not modified by the reference to a moving observer.

With the neutral wave speed thus determined, equation (55) can be integrated to give the amplitude of the oscillations. For this purpose, it is necessary to find the proper value for α. This can be done by several trials, with the first approximation given by the ratio of the integrals in equation (59). The ratio will yield the characteristic value only when the function $f(y)$ is the characteristic function, but it is known that any reasonable approximation to it will give a very close approximation to the characteristic value.

Calculations of these neutral oscillations are carried out for several Mach numbers in one stream with the other at rest. These are given in table V and figure 6.

SELF-EXCITED OSCILLATIONS

The formulas given in the section "General Stability Characteristics in the Inviscid Case" have been used to calculate the characteristics of self-excited oscillations. It is found that

$$\frac{dc}{d\alpha} = 0.177 - 0.209i$$

for the case $M_1 = 1$ and

$$\frac{dc}{d\alpha} = 0.093 - 0.287i$$

for the incompressible case. The results are shown diagrammatically in figure 7. It is seen that the extent of amplification is fairly large. There is also indicated a decrease of amplification with increasing Mach number.

It would be easy to calculate the amplitude function of these self-excited oscillations by using the characteristic values obtained above. This was not carried out because of limitations in time.

NATURE OF OSCILLATIONS IN LIMIT OF INFINITE REYNOLDS NUMBER

Calculations in the section "Neutral Oscillations" indicate that the neutral oscillation has quite a simple amplitude distribution. In fact, it does not show any node. The amplified disturbances are expected to show similar characteristics. The mathematical theory (references 5 and 6), however, indicates that the damped oscillations behave in a much more complicated manner. It is concluded that there is always a finite viscous region in the interior of the fluid, no matter how large the Reynolds number may be. In fact, a minimum width of this region is determined. This has to do with the crossing of the solid lines by the real axis of y in figure 5. The solution in the part of the real axis between the solid lines shows exponential behavior - and therefore viscous nature - if the solution in the outside parts shows the inviscid behavior. This type of conclusion has been reached in reference 1. However, it was possible only to suggest that such viscous behavior would occur at the solid lines. The improved theory shows that it must occur throughout the region in between. It is important to note that the complex conjugate of an amplified solution does not represent a damped oscillation and vice versa, although this conclusion can be easily reached by a cursory examination of the inviscid equations. The damped oscillations do not satisfy the inviscid equation all along the real axis; otherwise, they could not take on the proper branch of the logarithm as specified in the section "General Study of Small Disturbances in a Nearly Parallel Flow Field in a Compressible Fluid." They exhibit a behavior very much like that of the vorticity distribution in fully developed turbulence flow. For large Reynolds numbers, there is one part of the space where the vorticity is highly concentrated; in another part, there is very little vorticity. This illustrates the two kinds of limiting behavior of a viscous fluid in the limit of infinite Reynolds number: In one part of the field, the inviscid behavior is approached; in another part, it becomes highly oscillatory spatially.

CONCLUSIONS

From a study of the stability of the mixing of two parallel streams in a gas, the following conclusions may be drawn:

1. If the relative speed of the two parallel streams exceeds the sum of their velocities of sound, subsonic oscillations cannot occur, and the mixing region may be expected to be stable with respect to small disturbances.

2. A further necessary condition for the possible occurrence of small subsonic disturbances is that somewhere in the field

$$\frac{d}{dy}\left(\rho \frac{dw}{dy}\right) = 0$$

where y is a positional coordinate across the stream, ρ is the density of the gas, and w is a dimensionless velocity distribution. This condition is usually satisfied for the present class of problems.

3. If the speed of the flow at such a point is denoted by w_s, then the field of flow can execute a neutral wavy oscillation having a finite wave length and propagating with the speed $c = w_s$ if and only if w_s is subsonic relative to both streams. There is no other possible neutral oscillation. This leads to a more strict condition of stability than that given by conclusion 1.

4. Under the above conditions, the field of flow can execute amplified wavy oscillations having wave lengths longer than that of the neutral oscillation. Oscillations having shorter wave lengths are damped. (This specific form of the conclusion was not obtained for the boundary layer at a solid surface in NACA TN 1115, but its validity can be shown by the present method.) The extent of amplification in such cases is fairly large.

5. At large Reynolds numbers, the amplified disturbances are essentially free from the effect of viscosity. On the other hand, disturbances with finite damping are expected to exhibit a highly oscillatory behavior over a finite region in the field of flow. This is similar to the structure of the vorticity field in fully developed turbulence.

6. For the case of constant enthalpy with one stream at rest, the wave length of the neutral disturbances increases with increasing Mach

number in the other stream. The flow becomes completely stable at a free-stream Mach number of 1.7. This is more restrictive than the value 2.5 obtained by applying conclusion 1 to the present case.

Massachusetts Institute of Technology
Cambridge, Mass., July 30, 1952

APPENDIX

SYMBOLS

The quantities bearing subscript 1 in the last column are the dimensional quantities in the first free stream. Corresponding quantities in the second stream bear a subscript 2. The quantities without a prime satisfy the equations of steady motion; those with primes satisfy the disturbance equations.

Dimensional quantities	Dimensionless quantities	Reference quantities
Positional coordinates:		
x^*	x	$l = 2\sqrt{2\nu_1 x^*/U_1}$
y^*	$y = \eta/2\sqrt{2}$	l
Time:		
t^*	t	l/U_1
Velocity components in directions of x- and y-axes, respectively:		
$u^* + u^{*\prime}$	$w(y)+f(y)e^{i\alpha(x-ct)}$	U_1
$v^* + v^{*\prime}$	$-\alpha\phi(y)e^{i\alpha(x-ct)}$	U_1
Density of gas:		
$\rho^* + \rho^{*\prime}$	$\rho(y)+r(y)e^{i\alpha(x-ct)}$	ρ_1
Pressure of gas:		
$p^* + p^{*\prime}$	$p(y)+\pi(y)e^{i\alpha(x-ct)}$	p_1
Temperature of gas:		
$T^* + T^{*\prime}$	$T(y)+\theta(y)e^{i\alpha(x-ct)}$	T_1
Coefficient of viscosity of gas:		
μ^*	μ	μ_1

Dimensional quantities	Dimensionless quantities	Reference quantities
Thermal conductivity:		
k^*	μ/Pr	$c_p\mu_1$
Wave number of disturbance:		
$\alpha^* = 2\pi/\lambda^*$	$\alpha = 2\pi/\lambda$	l^{-1}
Phase velocity of disturbance:		
c^*	c	U_1
Specific heat at constant volume:		
c_v	1	c_v
Specific heat at constant pressure:		
c_p	γ	c_v
Gas constant per gram:		
R^*	$\gamma - 1$	c_v
Reynolds number		
	$R = \rho_1 U_1 l/\mu_1$	
Mach number		
	$M_1 = U_1\Big/\sqrt{\gamma R^* T_1}$	
Prandtl number		
	$Pr = c_p\mu^*/k^*$	

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TABLE I

VELOCITY DISTRIBUTIONS IN INCOMPRESSIBLE CASE

[For definition of symbols, see equations (10), (14), and (15b); $w = u^*/U_1$]

(a) For $\lambda = 0.2$ and $\theta = 0.098$

η	w	η	w	η	w
-9.0	0.667	-2.6	0.684	1.4	0.941
-8.8	-----	-2.4	.689	1.6	.952
-8.4	-----	-2.2	.695	1.8	.961
-8.0	-----	-2.0	.701	2.0	.969
-7.6	-----	-1.8	.709	2.2	.976
-7.2	-----	-1.6	.719	2.4	.982
-6.8	-----	-1.4	.729	2.6	.986
-6.4	-----	-1.2	.741	2.8	.990
-6.0	-----	-1.0	.754	3.0	.993
-5.6	-----	-.8	.768	3.2	.995
-5.2	.668	-.6	.784	3.4	.996
-4.8	.668	-.4	.800	3.6	.997
-4.4	.668	-.2	.816	3.8	.998
-4.0	.669	0	.833	4.0	.999
-3.8	.670	.2	.851	4.2	.999
-3.6	.671	.4	.868	4.4	.999
-3.4	.673	.6	.884	4.6	1.000
-3.2	.675	.8	.900	4.8	1.000
-3.0	.677	1.0	.915	5.0	1.000
-2.8	.680	1.2	.928		

TABLE I

VELOCITY DISTRIBUTIONS IN INCOMPRESSIBLE CASE - Continued

(b) For $\lambda = 0.4$ and $\theta = 0.310$

η	w	η	w	η	w
-9.0	0.429	-2.6	0.470	1.4	0.891
-8.8	------	-2.4	.479	1.6	.909
-8.4	------	-2.2	.489	1.8	.926
-8.0	------	-2.0	.501	2.0	.941
-7.6	------	-1.8	.515	2.2	.953
-7.2	------	-1.6	.531	2.4	.964
-6.8	------	-1.4	.548	2.6	.972
-6.4	------	-1.2	.567	2.8	.979
-6.0	------	-1.0	.588	3.0	.985
-5.6	.430	-.8	.611	3.2	.989
-5.2	.430	-.6	.635	3.4	.992
-4.8	.432	-.4	.661	3.6	.994
-4.4	.434	-.2	.687	3.8	.996
-4.0	.437	0	.714	4.0	.997
-3.8	.440	.2	.742	4.2	.998
-3.6	.443	.4	.769	4.4	.999
-3.4	.446	.6	.796	4.6	.999
-3.2	.451	.8	.822	4.8	1.000
-3.0	.456	1.0	.847	5.0	1.000
-2.8	.462	1.2	.870		

TABLE I

VELOCITY DISTRIBUTIONS IN INCOMPRESSIBLE CASE - Continued

(c) For $\lambda = 0.6$ and $\theta = 0.573$

η	w	η	w	η	w
-9.0	0.250	-2.6	0.319	1.4	0.848
-8.8	.250	-2.4	.331	1.6	.873
-8.4	.250	-2.2	.345	1.8	.896
-8.0	.250	-2.0	.361	2.0	.915
-7.6	.250	-1.8	.379	2.2	.933
-7.2	.250	-1.6	.398	2.4	.947
-6.8	.251	-1.4	.420	2.6	.959
-6.4	.251	-1.2	.444	2.8	.969
-6.0	.252	-1.0	.470	3.0	.977
-5.6	.253	-.8	.498	3.2	.983
-5.2	.255	-.6	.527	3.4	.988
-4.8	.258	-.4	.559	3.6	.991
-4.4	.263	-.2	.591	3.8	.994
-4.0	.270	0	.625	4.0	.996
-3.8	.274	.2	.659	4.2	.997
-3.6	.279	.4	.693	4.4	.998
-3.4	.285	.6	.727	4.6	.999
-3.2	.291	.8	.760	4.8	.999
-3.0	.299	1.0	.791	5.0	1.000
-2.8	.308	1.2	.821		

TABLE I

VELOCITY DISTRIBUTIONS IN INCOMPRESSIBLE CASE - Continued

(d) For $\lambda = 0.8$ and $\theta = 0.860$

η	w	η	w	η	w
-9.0	0.111	-2.6	0.209	1.4	0.813
-8.8	.111	-2.4	.224	1.6	.842
-8.4	.111	-2.2	.240	1.8	.869
-8.0	.111	-2.0	.258	2.0	.893
-7.6	.112	-1.8	.279	2.2	.914
-7.2	.112	-1.6	.301	2.4	.932
-6.8	.113	-1.4	.326	2.6	.947
-6.4	.115	-1.2	.352	2.8	.960
-6.0	.117	-1.0	.382	3.0	.970
-5.6	.120	-.8	.413	3.2	.977
-5.2	.124	-.6	.446	3.4	.984
-4.8	.129	-.4	.481	3.6	.988
-4.4	.137	-.2	.518	3.8	.992
-4.0	.146	0	.556	4.0	.994
-3.8	.152	.2	.594	4.2	.996
-3.6	.159	.4	.633	4.4	.998
-3.4	.167	.6	.672	4.6	.999
-3.2	.175	.8	.709	4.8	.999
-3.0	.185	1.0	.746	5.0	1.000
-2.8	.196	1.2	.780		

TABLE I

VELOCITY DISTRIBUTIONS IN INCOMPRESSIBLE CASE - Concluded

(e) For $\lambda = 1.0$ and $\theta = 1.160$

η	w	η	w	η	w
-9.0	0	-2.6	0.127	1.4	0.783
-8.8	.000	-2.4	.143	1.6	.816
-8.4	.001	-2.2	.161	1.8	.847
-8.0	.001	-2.0	.181	2.0	.874
-7.6	.002	-1.8	.202	2.2	.898
-7.2	.004	-1.6	.227	2.4	.919
-6.8	.006	-1.4	.253	2.6	.937
-6.4	.009	-1.2	.282	2.8	.951
-6.0	.013	-1.0	.313	3.0	.963
-5.6	.018	-.8	.347	3.2	.973
-5.2	.024	-.6	.383	3.4	.980
-4.8	.033	-.4	.420	3.6	.986
-4.4	.042	-.2	.459	3.8	.990
-4.0	.055	0	.500	4.0	.994
-3.8	.062	.2	.542	4.2	.996
-3.6	.070	.4	.584	4.4	.997
-3.4	.079	.6	.626	4.6	.998
-3.2	.089	.8	.668	4.8	.999
-3.0	.100	1.0	.708	5.0	1.000
-2.8	.113	1.2	.747		

TABLE II

VELOCITY AND TEMPERATURE DISTRIBUTIONS

IN COMPRESSIBLE CASE

[One stream at rest; $\lambda = 1$; $\theta = 1.160$]

(a) $M_1 = 0.5$

η	w	T	η	w	T
-12.6	0	1.050	-1.4	0.261	1.047
-12.2	-----	-----	-1.2	.289	1.046
-11.8	-----	-----	-1.0	.320	1.045
-11.4	-----	-----	-.8	.352	1.044
-11.0	-----	-----	-.6	.386	1.043
-10.6	-----	-----	-.4	.423	1.041
-10.2	-----	-----	-.2	.461	1.039
-9.8	-----	-----	0	.500	1.038
-9.4	0	1.050	.2	.540	1.035
-9.0	.001	-----	.4	.581	1.033
-8.6	.001	-----	.6	.622	1.031
-8.2	.002	-----	.8	.662	1.028
-7.8	.003	-----	1.0	.702	1.025
-7.4	.004	-----	1.2	.740	1.023
-7.0	.006	-----	1.4	.776	1.020
-6.6	.009	-----	1.6	.809	1.017
-6.2	.013	-----	1.8	.840	1.015
-5.8	.019	-----	2.0	.868	1.012
-5.4	.025	-----	2.2	.892	1.010
-5.0	.033	1.049	2.4	.914	1.008
-4.6	.043	-----	2.6	.932	1.007
-4.2	.054	-----	2.8	.947	1.005
-3.8	.069	-----	3.0	.960	1.004
-3.4	.086	-----	3.4	.978	1.002
-3.0	.108	-----	3.8	.989	1.001
-2.8	.121	-----	4.2	.995	1.001
-2.6	.136	-----	4.6	.998	1.000
-2.4	.152	-----	5.0	.999	1.000
-2.2	.170	-----	5.4	1.000	1.000
-2.0	.190	1.048	5.6	1.000	1.000
-1.8	.211	1.048	6.0	1.000	1.000
-1.6	.235	1.047			

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TABLE II

VELOCITY AND TEMPERATURE DISTRIBUTIONS

IN COMPRESSIBLE CASE - Continued

(b) $M_1 = 1.0$

η	w	T	η	w	T
-12.6	0	1.200	-1.4	0.282	1.184
-12.2	-----	-----	-1.2	.309	1.181
-11.8	-----	-----	-1.0	.336	1.177
-11.4	-----	-----	-.8	.366	1.173
-11.0	-----	-----	-.6	.397	1.168
-10.6	-----	-----	-.4	.430	1.163
-10.2	0	1.200	-.2	.465	1.157
-9.8	.001	-----	0	.500	1.150
-9.4	.002	-----	.2	.537	1.142
-9.0	.002	-----	.4	.574	1.134
-8.6	.004	-----	.6	.611	1.125
-8.2	.005	-----	.8	.648	1.116
-7.8	.007	-----	1.0	.685	1.106
-7.4	.010	-----	1.2	.721	1.096
-7.0	.014	-----	1.4	.756	1.086
-6.6	.019	-----	1.6	.789	1.076
-6.2	.024	-----	1.8	.820	1.066
-5.8	.031	-----	2.0	.848	1.056
-5.4	.039	-----	2.2	.874	1.047
-5.0	.048	-----	2.4	.897	1.039
-4.6	.059	-----	2.6	.918	1.032
-4.2	.072	1.199	2.8	.935	1.025
-3.8	.088	1.198	3.0	.950	1.020
-3.4	.108	1.197	3.4	.971	1.011
-3.0	.132	1.197	3.8	.985	1.006
-2.8	.145	1.196	4.2	.993	1.003
-2.6	.160	1.195	4.6	.997	1.001
-2.4	.177	1.194	5.0	.999	1.001
-2.2	.195	1.192	5.4	1.000	1.000
-2.0	.214	1.191	5.6	-----	-----
-1.8	.235	1.189	6.0	-----	-----
-1.6	.258	1.187			

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TABLE II

VELOCITY AND TEMPERATURE DISTRIBUTIONS

IN COMPRESSIBLE CASE - Concluded

(c) $M_1 = 1.5$

η	w	T	η	w	T
-12.6	0	1.450	-1.4	0.310	1.407
-12.2	.001	------	-1.2	.334	1.400
-11.8	.001	------	-1.0	.359	1.392
-11.4	.002	------	-.8	.385	1.384
-11.0	.002	------	-.6	.412	1.374
-10.6	.003	------	-.4	.440	1.363
-10.2	.004	------	-.2	.470	1.351
-9.8	.006	------	0	.500	1.338
-9.4	.007	------	.2	.531	1.323
-9.0	.009	------	.4	.564	1.307
-8.6	.012	------	.6	.596	1.290
-8.2	.016	------	.8	.629	1.272
-7.8	.020	------	1.0	.662	1.253
-7.4	.025	------	1.2	.694	1.233
-7.0	.030	------	1.4	.727	1.213
-6.6	.037	1.449	1.6	.758	1.192
-6.2	.044	1.449	1.8	.788	1.171
-5.8	.052	1.449	2.0	.817	1.150
-5.4	.062	1.448	2.2	.844	1.130
-5.0	.073	1.448	2.4	.869	1.111
-4.6	.086	1.447	2.6	.891	1.093
-4.2	.102	1.445	2.8	.911	1.076
-3.8	.120	1.444	3.0	.929	1.062
-3.4	.142	1.441	3.4	.957	1.038
-3.0	.167	1.438	3.8	.976	1.021
-2.8	.180	1.435	4.2	.988	1.011
-2.6	.196	1.433	4.6	.994	1.005
-2.4	.212	1.430	5.0	.998	1.002
-2.2	.229	1.426	5.4	.999	1.001
-2.0	.248	1.422	5.6	1.000	1.000
-1.8	.267	1.418	6.0	1.000	1.000
-1.6	.288	1.413			

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TABLE III

WAVE SPEEDS OF POSSIBLE NEUTRAL SUBSONIC DISTURBANCES

[Values of $w_s = u^*/U_1$ defined by equation (70)]

M_1	λ				
	0.2	0.4	0.6	0.8	1.0
0	0.834	0.732	0.657	0.613	0.576
.5	.845	.743	.678	.632	.594
1	.851	.762	.704	.664	.633
2	.878	.823	.796	.772	.762
5	.949	.940	.941	.938	.938
10	.984	.981	.981	.982	.982

TABLE IV

CONDITIONS ASSOCIATED WITH NEUTRAL SONIC DISTURBANCE

Conditions	λ				
	0.2	0.4	0.6	0.8	1.0
M_1	>10	4.0	2.7	2.0	1.7
M_2	>1.92	.90	.44	.167	0
c	≈1	.91	.82	.78	.73

TABLE V

WAVE NUMBER AND AMPLITUDE OF NEUTRAL OSCILLATIONS

$$\left[y = \eta/2\sqrt{2}, \quad \theta = 1.160\right]$$

(a) Wave number of neutral oscillations for various Mach numbers

M_1	Wave number, $\alpha\theta$
0	0.459
.5	.454
1.0	.374
1.5	.324

TABLE V

WAVE NUMBER AND AMPLITUDE OF NEUTRAL OSCILLATIONS - Continued

(b) Amplitude of neutral oscillations for $M_1 = 0$

y	ϕ	$d\phi/dy$	y	ϕ	$d\phi/dy$	y	ϕ	$d\phi/dy$
-6.4	0.001	0.001	-1.0	0.297	0.269	2.2	0.107	-0.107
-6.2	.001	.001	-.9	.325	.281	2.4	.087	-.087
-6.0	.001	.001	-.8	.354	.291	2.6	.071	-.071
-5.8	.002	.002	-.7	.383	.296	2.8	.058	-.058
-5.6	.002	.002	-.6	.412	.294	3.0	.048	-.048
-5.4	.002	.003	-.5	.442	.286	3.2	.039	-.039
-5.2	.003	.003	-.4	.469	.269	3.4	.032	-.032
-5.0	.004	.004	-.3	.495	.243	3.6	.026	-.026
-4.8	.005	.005	-.2	.518	.206	3.8	.022	-.022
-4.6	.006	.007	-.1	.536	.159	4.0	.018	-.018
-4.4	.008	.008	0	.549	.103	4.2	.014	-.014
-4.2	.009	.011	.1	.556	.040	4.4	.012	-.012
-4.0	.012	.013	.2	.557	-.028	4.6	.010	-.010
-3.8	.015	.016	.3	.550	-.097	4.8	.008	-.008
-3.6	.018	.021	.4	.537	-.162	5.0	.007	-.007
-3.4	.023	.026	.5	.518	-.220	5.2	.005	-.005
-3.2	.029	.032	.6	.494	-.267	5.4	.004	-.004
-3.0	.036	.040	.7	.465	-.303	5.6	.004	-.004
-2.8	.045	.050	.8	.434	-.325	5.8	.003	-.003
-2.6	.056	.062	.9	.401	-.334	6.0	.002	-.002
-2.4	.070	.077	1.0	.367	-.332	6.2	.002	-.002
-2.2	.087	.095	1.1	.335	-.322	6.4	.002	-.002
-2.0	.108	.116	1.2	.303	-.305	6.6	.001	-.001
-1.9	.120	.129	1.3	.274	-.283	6.8	.001	-.001
-1.8	.134	.142	1.4	.247	-.260	7.0	.001	-.001
-1.7	.149	.156	1.5	.222	-.237			
-1.6	.165	.172	1.6	.199	-.213			
-1.5	.183	.188	1.7	.179	-.192			
-1.4	.202	.204	1.8	.161	-.171			
-1.3	.224	.221	1.9	.145	-.152			
-1.2	.247	.238	2.0	.131	-.135			
-1.1	.271	.254	2.1	.118	-.113			

TABLE V

WAVE NUMBER AND AMPLITUDE OF NEUTRAL OSCILLATIONS - Concluded

(c) Amplitude of neutral oscillations for $M_1 = 1$

y	ϕ	$d\phi/dy$	y	ϕ	$d\phi/dy$	y	ϕ	$d\phi/dy$
-7.4	0.003	0.003	-0.3	0.551	0.284	2.9	0.122	-0.106
-7.0	.005	.003	-.2	.579	.273	3.0	.112	-.098
-6.6	.006	.005	-.1	.605	.254	3.1	.103	-.090
-6.2	.008	.006	0	.629	.227	3.2	.094	-.082
-5.8	.011	.008	.1	.650	.190	3.3	.087	-.076
-5.4	.015	.011	.2	.667	.142	3.4	.079	-.070
-5.0	.020	.015	.3	.678	.085	3.6	.067	-.059
-4.6	.026	.020	.4	.684	.020	3.8	.056	-.049
-4.2	.035	.026	.5	.682	-.050	4.0	.048	-.042
-3.8	.048	.035	.6	.674	-.121	4.2	.040	-.035
-3.4	.064	.047	.7	.658	-.189	4.4	.034	-.030
-3.0	.086	.063	.8	.636	-.249	4.6	.029	-.025
-2.6	.115	.084	.9	.609	-.297	4.8	.024	-.021
-2.2	.153	.112	1.0	.577	-.331	5.0	.020	-.018
-2.0	.177	.129	1.1	.543	-.351	5.2	.017	-.015
-1.9	.191	.138	1.2	.507	-.359	5.4	.014	-.013
-1.8	.205	.148	1.3	.471	-.356	5.6	.012	-.011
-1.7	.220	.158	1.4	.436	-.344	5.8	.010	-.009
-1.6	.237	.169	1.5	.403	-.328	6.0	.009	-.008
-1.5	.254	.180	1.6	.371	-.308	6.2	.007	-.006
-1.4	.273	.192	1.7	.341	-.288			
-1.3	.292	.204	1.8	.3.3	-.267			
-1.2	.313	.217	1.9	.288	-.246			
-1.1	.336	.229	2.0	.264	-.227			
-1.0	.359	.242	2.1	.242	-.209			
-.9	.384	.254	2.2	.222	-.192			
-.8	.410	.265	2.3	.204	-.177			
-.7	.437	.275	2.4	.187	-.163			
-.6	.465	.282	2.5	.171	-.150			
-.5	.493	.287	2.6	.157	-.137			
-.4	.522	.288	2.7	.144	-.126			
			2.8	.132	-.116			

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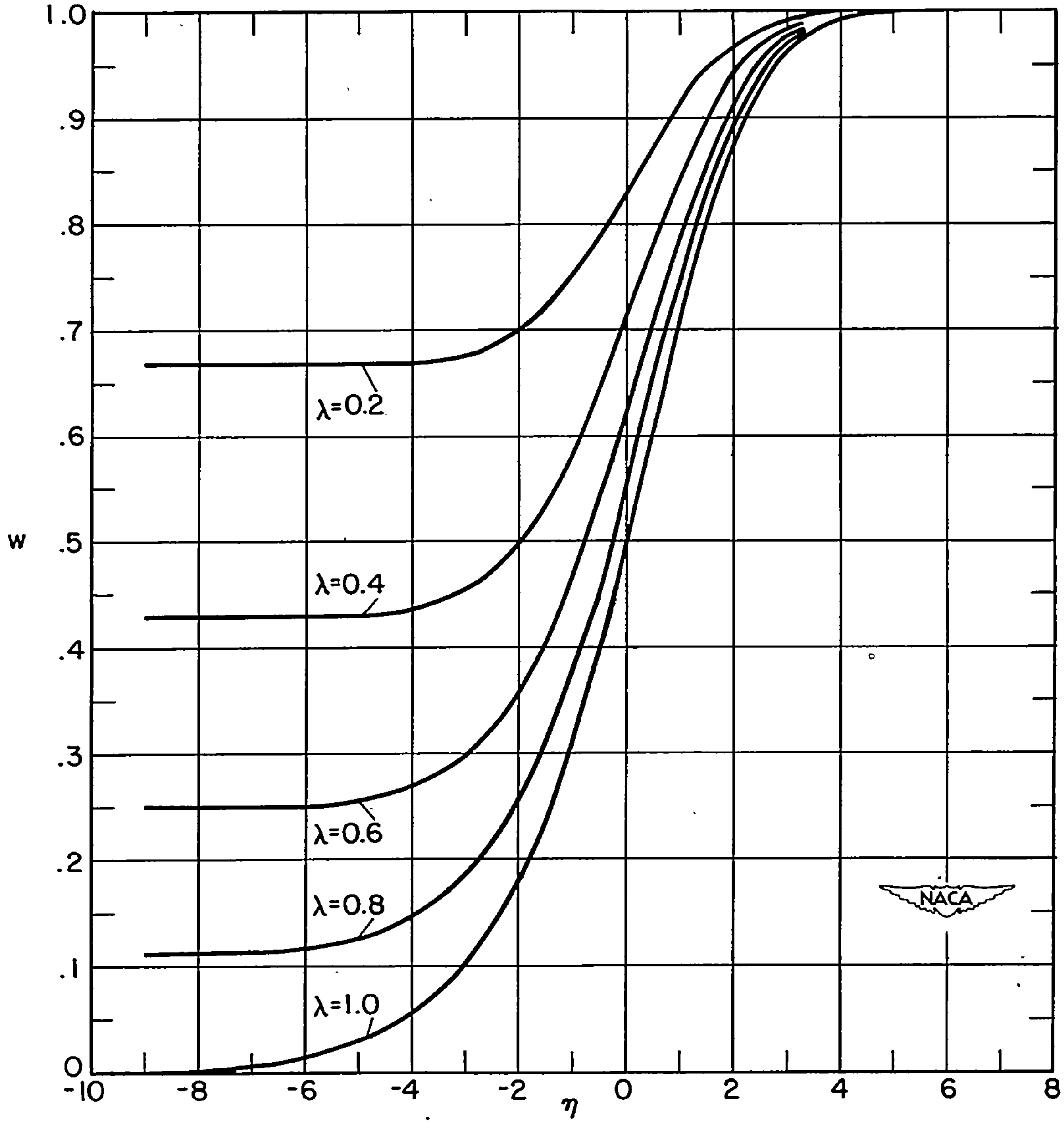
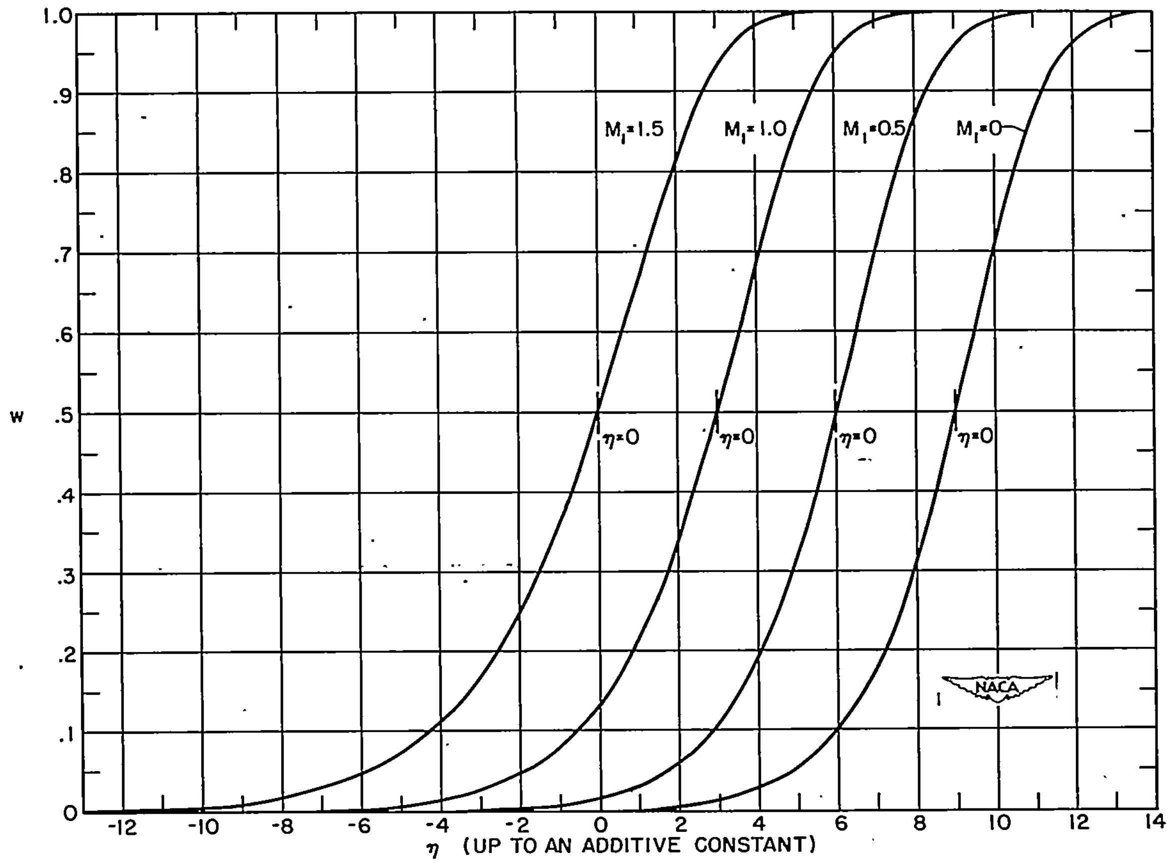


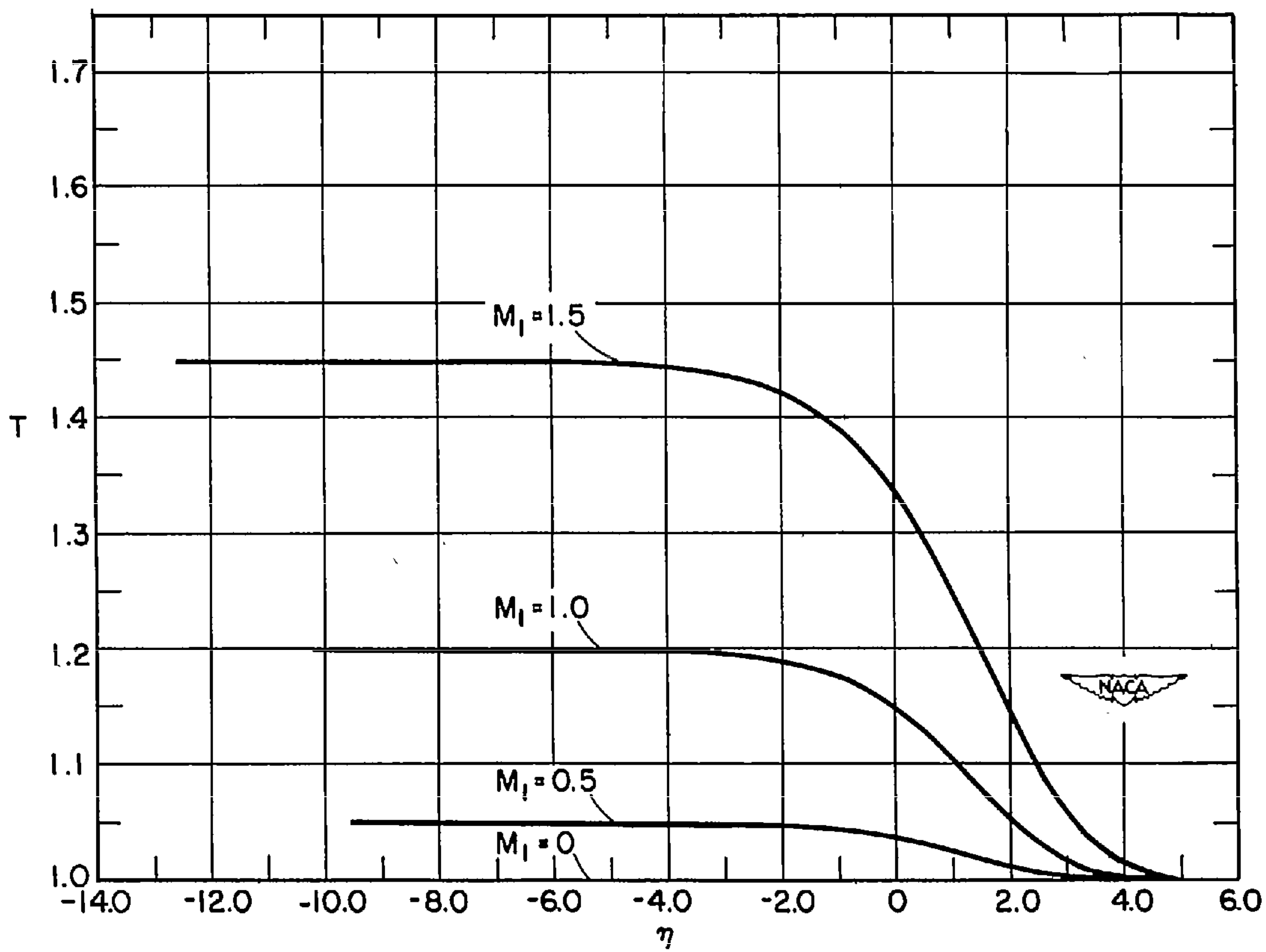
Figure 1.- Velocity distributions in the incompressible case. $w = \frac{U^*}{U_1}$;

$$\eta = \frac{y^*}{\sqrt{\nu_1 x^*/U_1}}; \ \lambda = \frac{U_1 - U_2}{U_1 + U_2}.$$



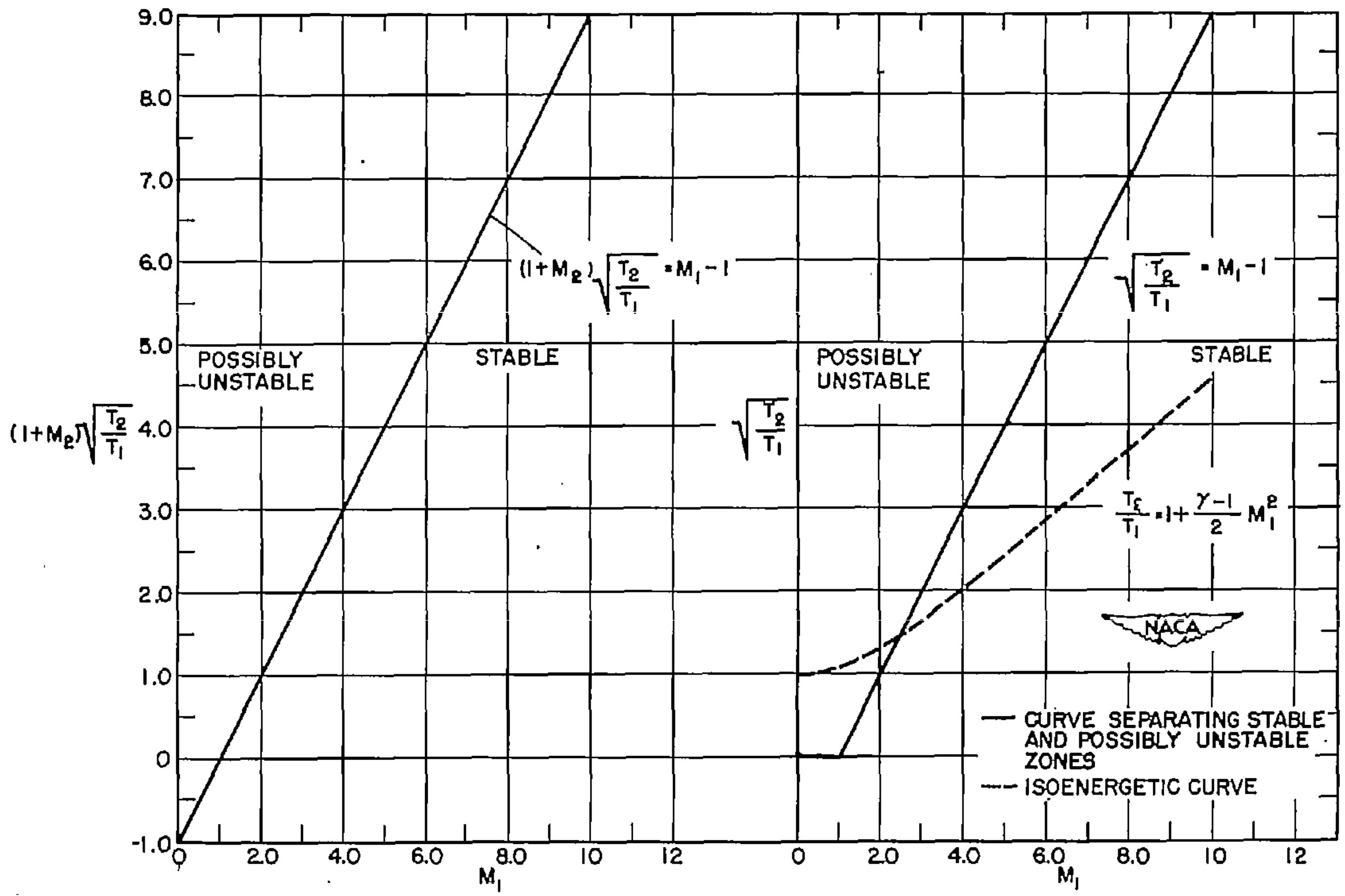
(a) Velocity distributions. $w = U^*/U_1$.

Figure 2.- Velocity and temperature distributions in compressible case. $\eta = \dfrac{y^*}{\sqrt{\nu_1 x^*/U_1}}$; $\lambda = \dfrac{U_1 - U_2}{U_1 + U_2} = 1.0$; M_1, Mach number of moving stream. $M_2 = 0$.



(b) Temperature distributions. $T = T^*/T_1$.

Figure 2.- Concluded.



(a) With both streams moving. M_1 and M_2, Mach numbers of two streams.

(b) With one stream at rest. M_1, Mach number of moving stream; $M_2 = 0$; $\gamma = 1.40$.

Figure 3.- Stable zones for various temperatures and Mach numbers. T_2/T_1, ratio of absolute temperatures of two streams.

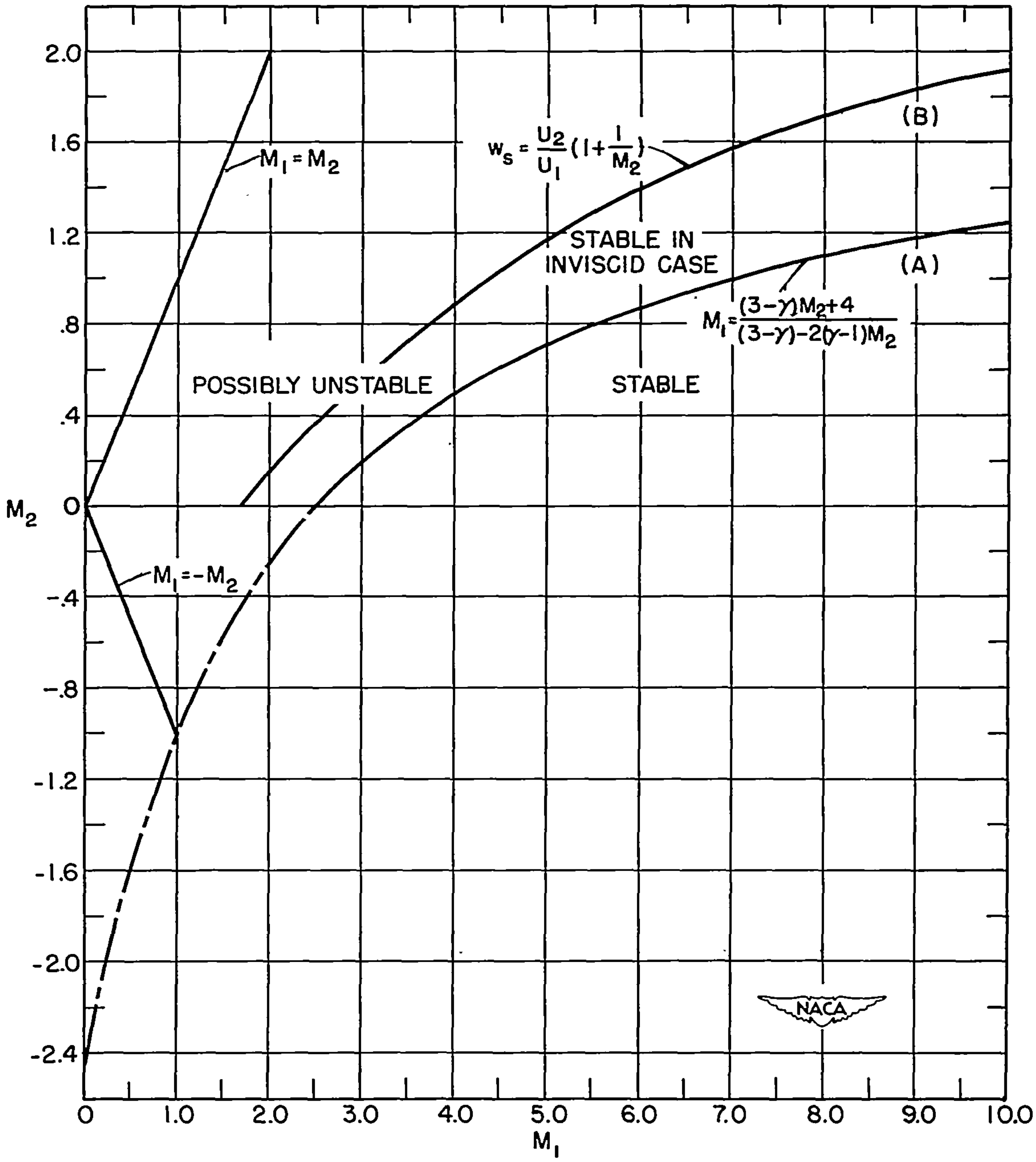


Figure 4.- Stable zones for isoenergetic case. $\gamma = 1.40$; M_1 and M_2, Mach numbers of two streams; w_s, wave speed of neutral disturbance; U_1 and U_2, velocities of two streams.

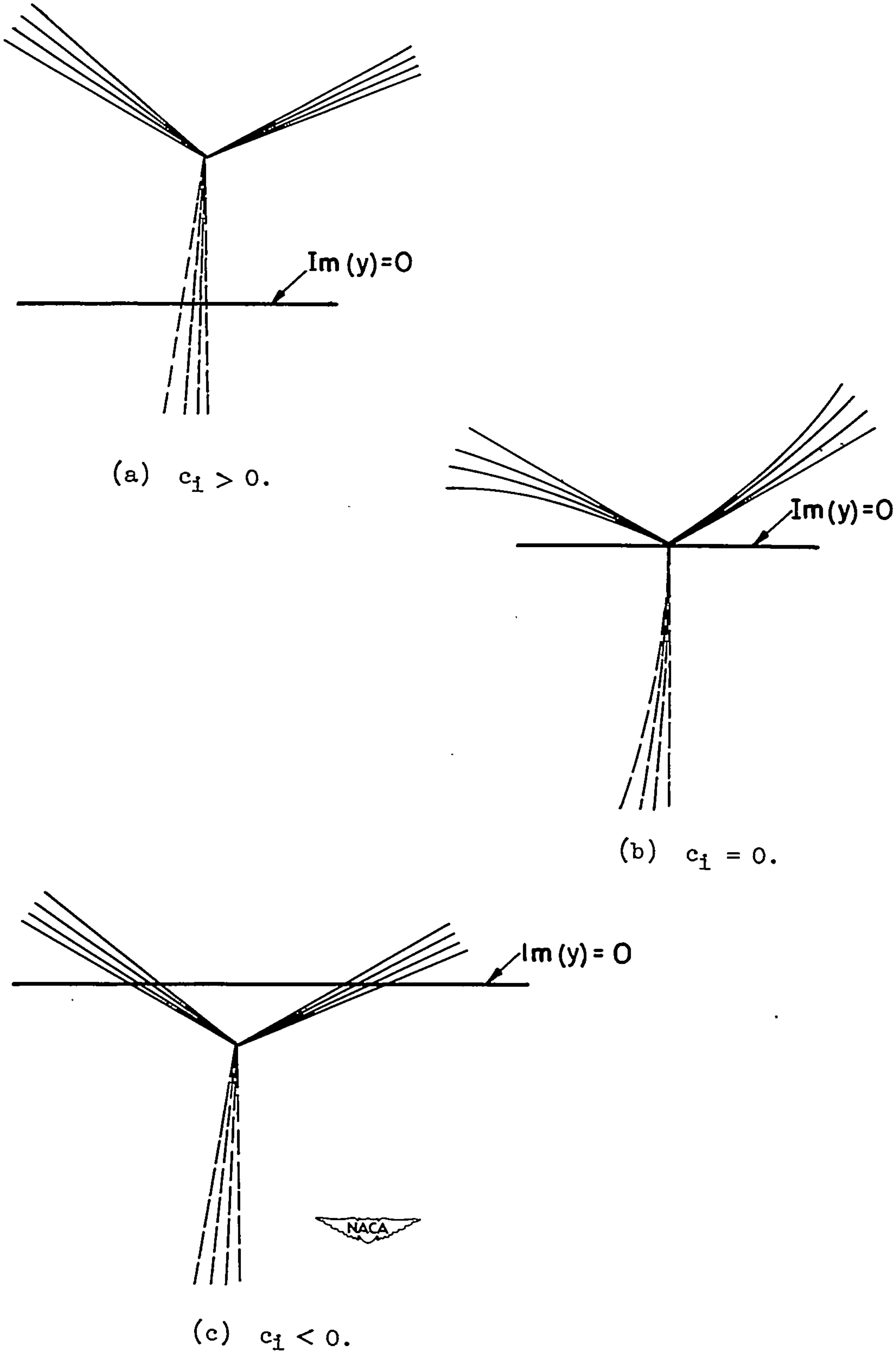
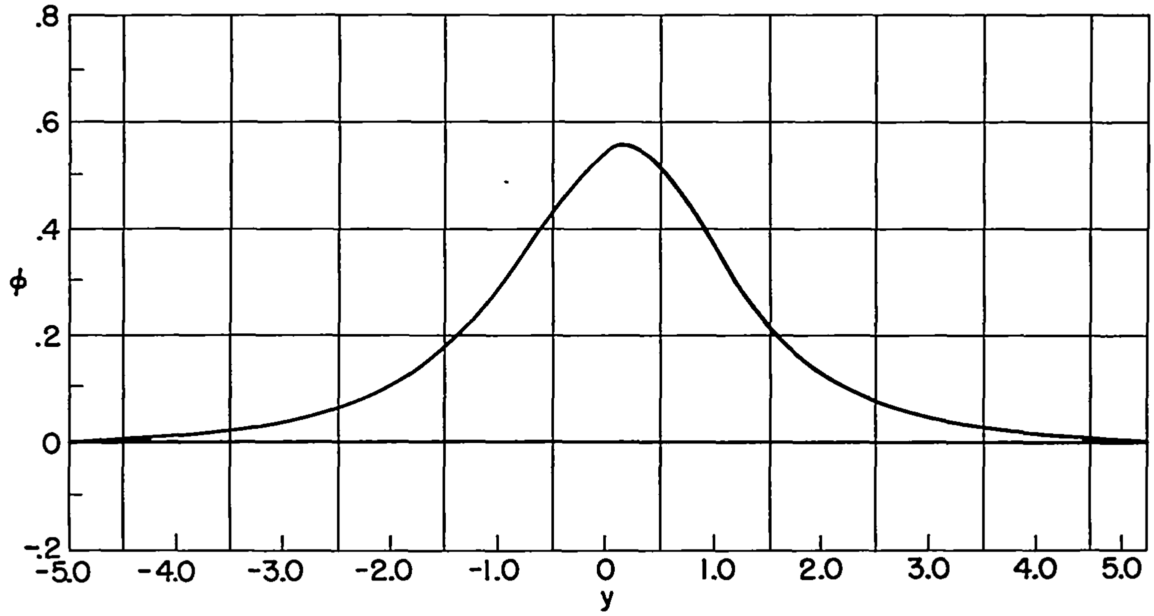
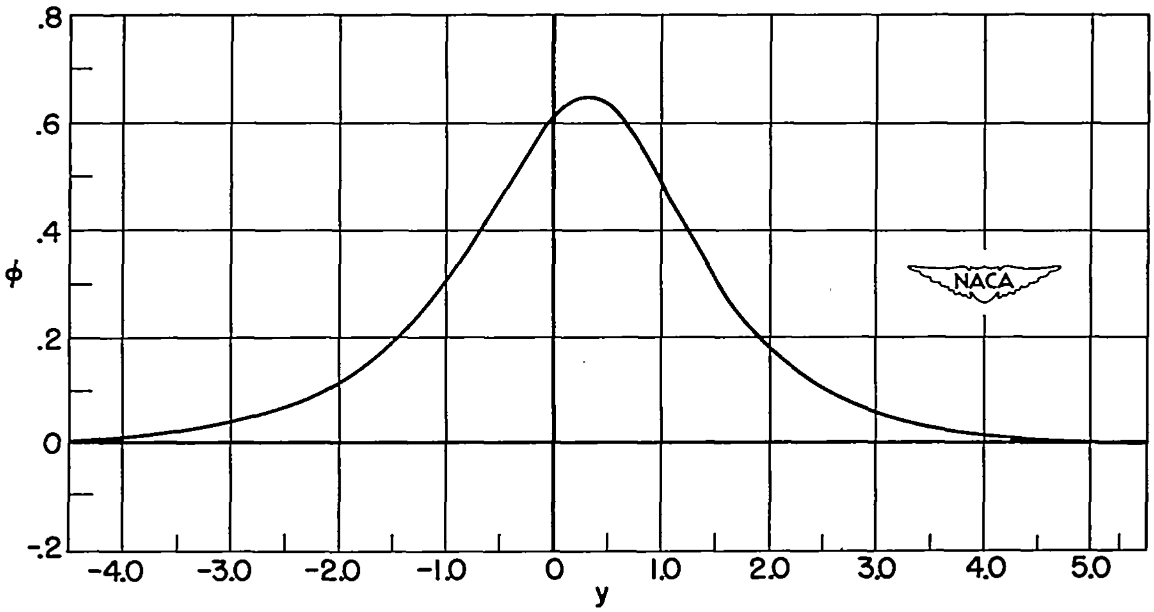


Figure 5.- Geometry of critical curves for asymptotic solutions.

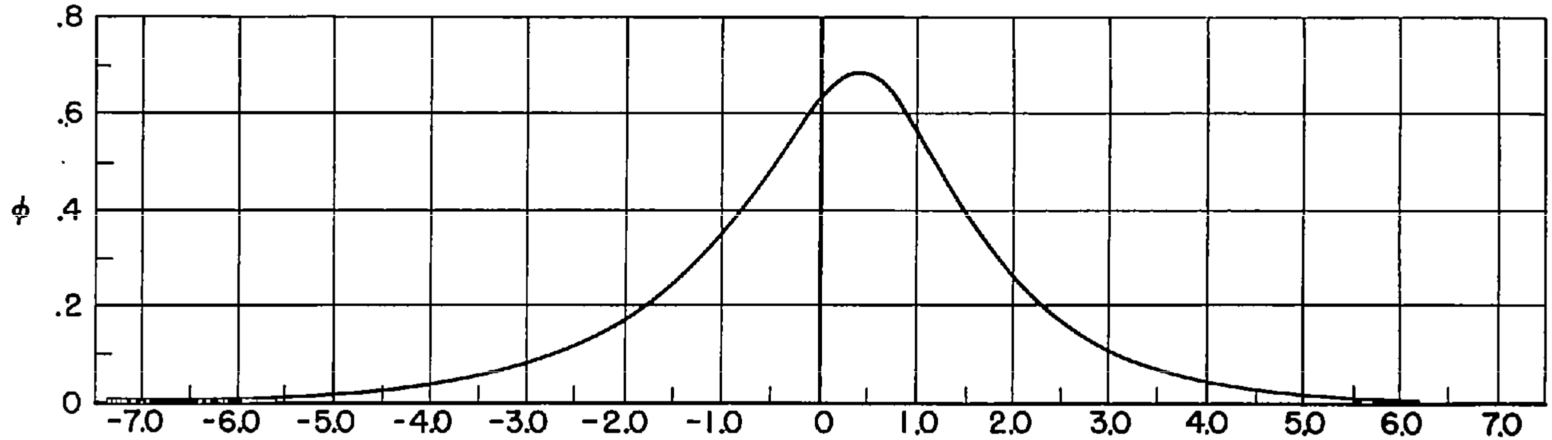


(a) For $M_1 = 0$ and $\alpha\theta = 0.459$.

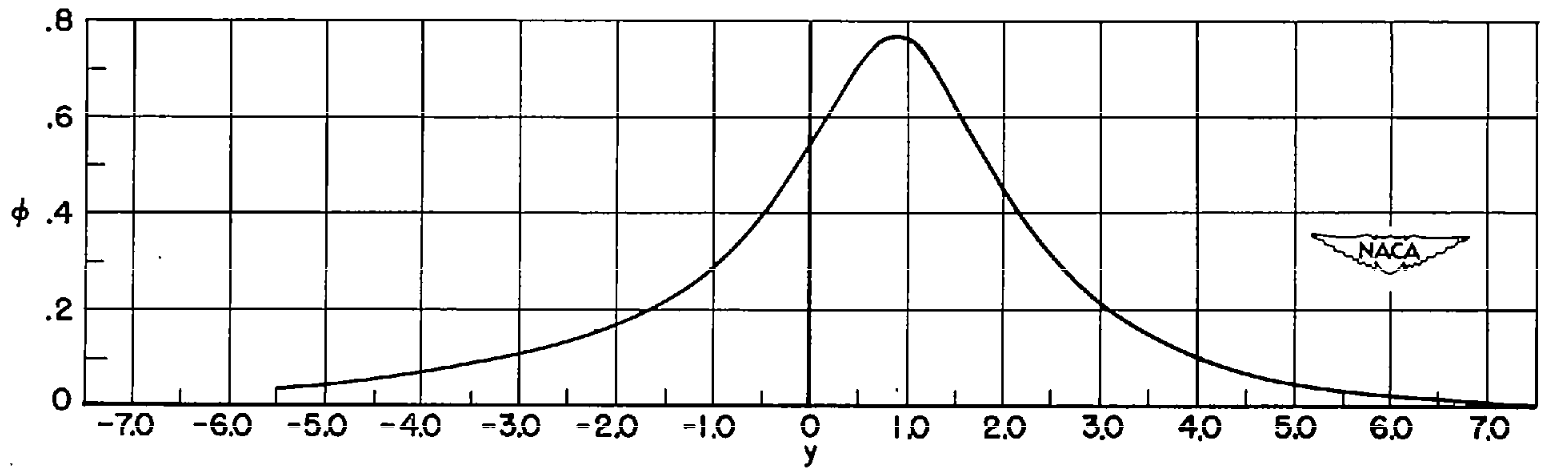


(b) For $M_1 = 0.5$ and $\alpha\theta = 0.454$.

Figure 6.- Amplitude of neutral oscillations. $y = \frac{1}{2\sqrt{2}} \eta = \frac{1}{2\sqrt{2}} \frac{y^*}{\sqrt{\nu_1 x^* / U_1}}$; ϕ, amplitude; α, wave number; θ, momentum-boundary-layer thickness; $\theta = 1.160$ in η-units.

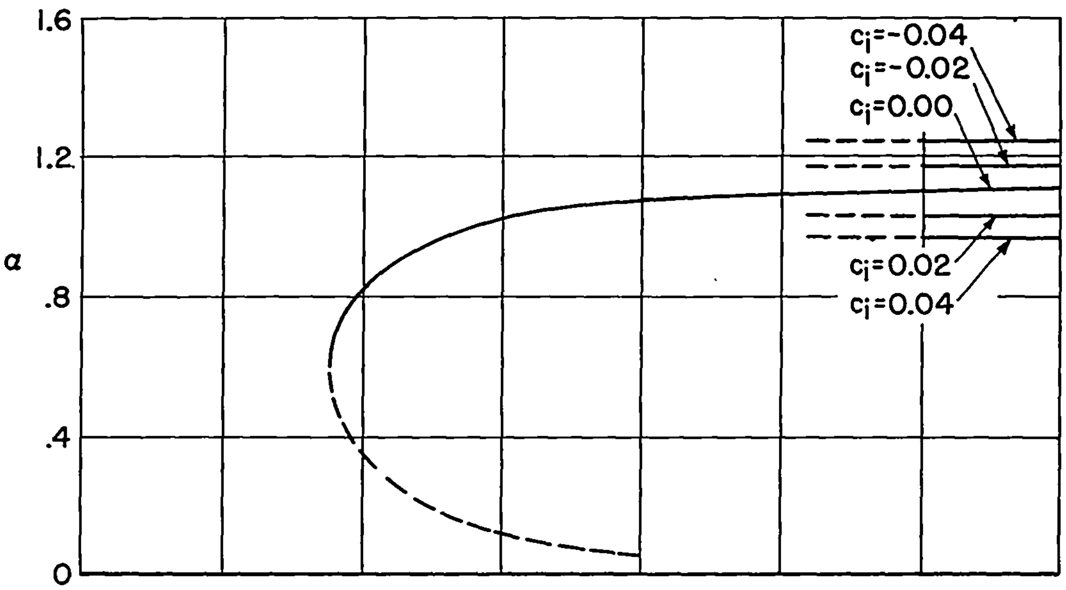


(c) For $M_1 = 1.0$ and $\alpha\theta = 0.374$.

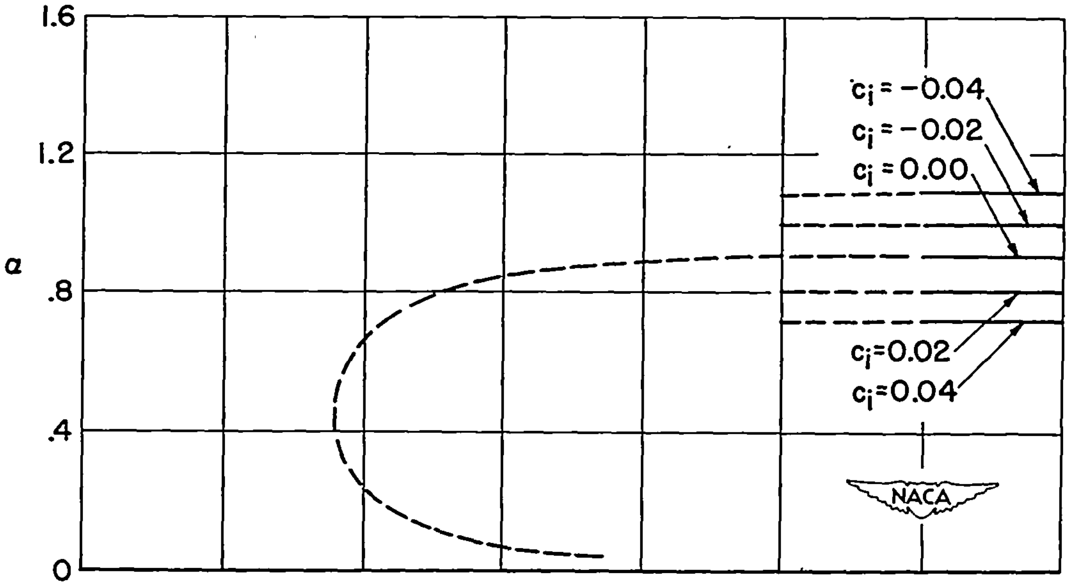


(d) For $M_1 = 1.5$ and $\alpha\theta = 0.324$.

Figure 6.- Concluded.



(a) For $M_1 = 0$.



(b) For $M_1 = 1$.

Figure 7.- Extent of amplification at infinite Reynolds number. α, wave number; $\frac{2\pi}{\alpha}$, wave length in y-units; $y = \frac{1}{2\sqrt{2}} \eta = \frac{1}{2\sqrt{2}} \frac{y^*}{\sqrt{\nu_1 x^*/U_1}}$; R, Reynolds number; c, dimensionless complex wave speed $\left(c_r + ic_i\right)$.

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